Subject: Mathematics
Lesson: Concavity, Points of Inflexion, Curve Sketching
Course Developer: Dr. Sada Nand Prasad
College/Department: A.N.D. College (D.U.)

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Table of Contents:

1. Learning Outcomes:

After studying this chapter, you should be able to

- apply second derivative to determine the behaviour of any curve in a given domain.
- compute the range for which any function is convex or concave or having points of inflexion;
- determine whether the curve is concave up or concave down;
- obtain the points of inflexion of a curve;
- identify and draw the graphs of some significant curves;

2. Introduction:

Application of differentiation is very useful in determining the solution of problems, we often face in almost all branches of science, like how to get accurate values of any function corresponding to any given values, how to find the maximum and minimum values of any function in a certain domain, how to determine the behaviour of any curve in a given domain etc. One of the ways of determining the behaviour of a curve is finding concavity and points of inflexion of the curve.

In this chapter we shall explain the process of finding the concavity, convexity and points of inflexion of a function. We shall explain the methods of tracing a given curve. To start with, we have talked about the problem of finding concavity, convexity and points of inflexion, which are geometrical applications of differentiation.

3. Concavity:

Let P be a given point on a curve. Draw the tangent to the curve at the point P. Let L be a given straight line and let θ be the acute angle formed by the tangent at P with the line L.

Definition (Concavity at a Point): The curve is said to be concave at P with respect to line L if a sufficiently small arc containing P, on extending to both sides of P lies entirely within the angle of θ.(Fig 1)

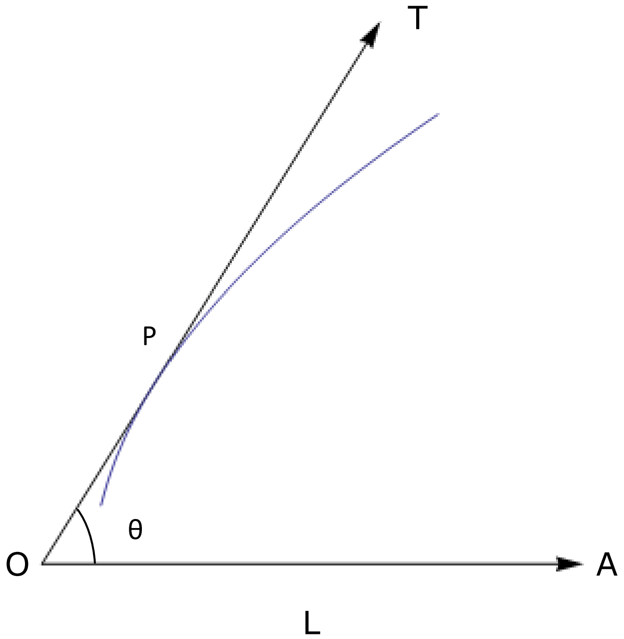


Fig 1: A curve concave or concave downwards at P

Definition (Convexity at a Point): The curve is said to be convex at P with respect to line L if a sufficiently small arc containing P, on extending to both sides of P lies entirely outside the angle of θ.(Fig 2).

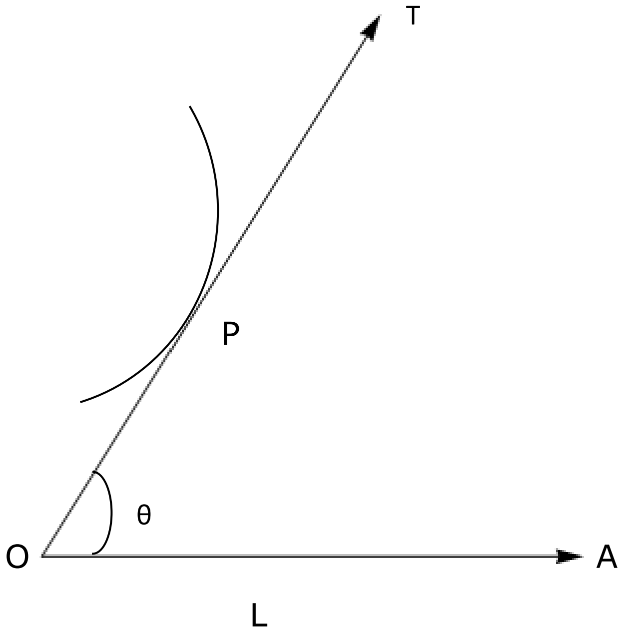


Fig 2: A curve convex or concave upwards at P

Value Additions:
(1) If $f''(x)>0,$ at every point of the arc, then the arc is concave up. For $\frac{d}{dx}\{f'(x)\}>0,$ gradient of the curve is an increasing function. (2) If $f''(x)<0,$ at every point of the arc, then the arc is concave down or convex up since the gradient of the curve is a decreasing function. (3) The curve is convex or concave at a point P with respect to the x – axis according as $y\frac{d^2y}{dx^2}$ is positive or negative at P.

Example 1: Show that the curve $y=e^x$ is convex everywhere.

Solution: We have,

$$y=e^x,$$

Differentiating the equation w.r.t. x we get

$$\frac{dy}{dx}=e^x,$$

Again differentiating w.r.t. x we get

$$\frac{d^2y}{dx^2}=e^x,$$

$$\Rightarrow \qquad y\frac{d^2y}{dx^2}=e^x.e^x=\left(e^x\right)^2>0$$

Since $y\dfrac{d^2y}{dx^2}$ is positive for all values of x, the curve is at every point convex to the foot of the corresponding ordinate.

Example 2: Find the range of values of x for which the curve $y=x^4-6x^3+12x^2+5x-9$ is concave upwards or downwards.

Solution: Given curve is

$$y=x^4-6x^3+12x^2+5x-9$$

Differentiating twice with respect to x, we get

$$\frac{d^2y}{dx^2}=12x^2-36x+24$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2}=12(x-1)(x-2)\begin{cases}>0, & if\ x>2 \text{ or } x<1\\ <0, & if\ 1<x<2\\ =0, & if\ x=1 or\ 2\end{cases}$$

Differentiating once again with respect to x, we get

$$\frac{d^3y}{dx^3}=24x-36=12(2x-3)$$

Hence the curve is concave upward in the interval $(-\infty,1)\cup(2,\infty)$ and concave downward in the interval $(1,2)$.

Example 3: Find the range of values of x for which the curve $y=x^3-x$ is concave upwards or downwards.

Solution: We have,

$$y=x^3-x,$$

On differentiating w.r.t. x we have

$$\frac{dy}{dx}=3x^2-1,$$

Again differentiating w.r.t. x we have

$$\frac{d^2y}{dx^2}=6x\begin{cases}>0 \;\; if \;\; x>0 \\ <0 \;\; if \;\; x<0\end{cases}$$

Hence the curve is concave upwards for x > 0 and concave downwards for x < 0.

Example 4: Show that the curve $y=a\sin x+b\cos x$ is concave downwards for all points above x – axis.

Solution: Given curve is

$$y=a\sin x+b\cos x$$

On differentiating w.r.t. x we have

$$\frac{dy}{dx}=a\cos x-b\sin x$$

Again differentiating w.r.t. x we have

$$\frac{d^2y}{dx^2}=-a\sin x-b\cos x$$

$$\Rightarrow \qquad y\frac{d^2y}{dx^2}=-\left(a\sin x+b\cos x\right).\left(a\sin x+b\cos x\right)=-\left(a\sin x+b\cos x\right)^2<0$$

Hence the curve is concave downwards for all points above x – axis.

I.Q. 1

I.Q. 2

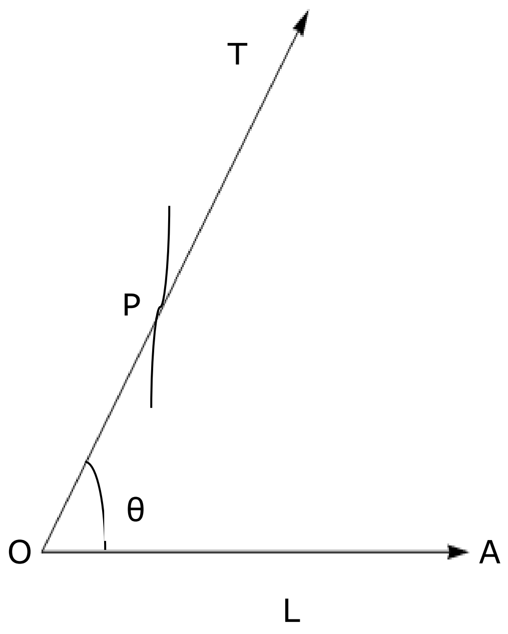
I.Q. 3

I.Q. 4

I.Q. 5

4. Points of inflexion:

Definition: The curve is said to have a point of inflexion if the arc of the curve on one side of P lies entirely within θ and that on the other side lies entirely outside the angle θ.(Fig 3)



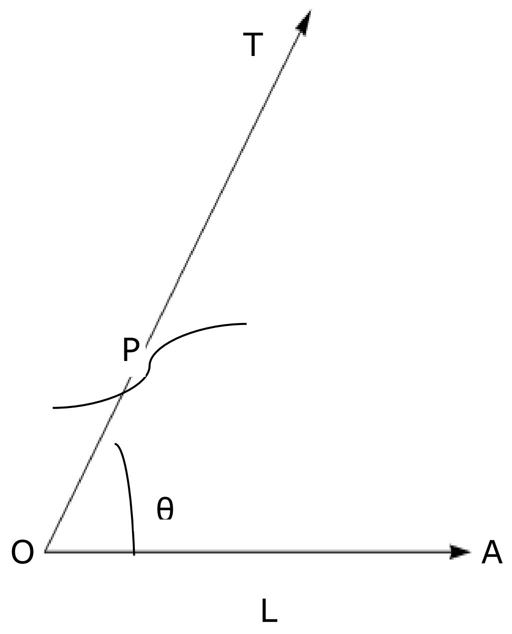


Fig 3: A curve having a point of inflexion at P

Value Addition:
(1) The points at which $f''(x)=0$ and the sign of $f''(x)$ changes about that point also the curve changes the direction of its concavity and crosses its tangent. Such points are called **points of inflexion** (See figure 3). At the points of inflexion $f''(x)=0 \, and \, f'''(x)\neq 0.$ (2) If $f''(x)=f'''(x)...=f^{n-1}(x)=0 \, and \, f^{n}(x)\neq 0,$ if n is even then the curve $y=f(x)$ is convex or concave at P with respect to the x-axis according as $f^{n}(x)>0 \, or \, f^{n}(x)<0,$, if n is odd then the curve has a point of inflexion at P.

Example 5: Investigate the point of inflexion on the curve $\left(a^2+x^2\right)y=a^2 x.$

Solutions: We have

$$\left(a^2+x^2\right)y=a^2 x \Rightarrow y=\frac{a^2 x}{\left(a^2+x^2\right)}$$

Differentiating twice with respect to x, we get

$$\frac{d^2y}{dx^2}=\frac{2a^2 x\left(x^2-3a^2\right)}{\left(a^2+x^2\right)^3}$$

Differentiating once again with respect to x, we get

$$\frac{d^3y}{dx^3}=\frac{2a^2\left[\left(a^2+x^2\right)^3\left(3x^2-3a^2\right)-x\left(x^2-3a^2\right)3\left(a^2+x^2\right)^2 2x\right]}{\left(a^2+x^2\right)^4}$$

$$=\frac{2a^2\left[3\left(a^2+x^2\right)\left(x^2-a^2\right)-6x^2\left(x^2-3a^2\right)\right]}{\left(a^2+x^2\right)^4}$$

$$\frac{d^2y}{dx^2}=0, and\ \frac{d^3y}{dx^3}\neq 0, when\ x=0\ and\ x=\pm a\sqrt{3}.$$

Hence the points of inflexion are $(0,0),\left(a\sqrt{3},\frac{a\sqrt{3}}{4}\right) and \left(-a\sqrt{3},\frac{-a\sqrt{3}}{4}\right)$

Example 6: In example 2, we find the range of values of x for the curve to be concave up or down. Find the points of inflexion for the same curve.

Solution: In example 2, differentiating the curve $y=x^4-6x^3+12x^2+5x-9$, thrice, we get, $\frac{d^3y}{dx^3}\neq 0, when\ x=1 or\ x=2.$ Hence points of inflexion at $x=1 or\ x=2$ and the points of inflexions are $(1,3),\ and\ (2,17)$

Example 7: Find the points of inflexion for the curve $y=\cos x, x\in(0,2\pi)$.

Solutions: We have

$$y=\cos x, x\in(0,2\pi),$$

Differentiating the equation w.r.t. x we have

$$\frac{dy}{dx}=-\sin x,$$

Differentiating the equation again w.r.t. x we have

$$\frac{d^2y}{dx^2}=-\cos x$$

on equating it equals to zero and solve for x, we have

$$\frac{d^2y}{dx^2}=0$$

$$\Rightarrow \qquad x=\frac{\pi}{2},\frac{3\pi}{2}\in(0,2\pi)$$

Also, we have

$$\frac{d^3y}{dx^3} = \sin x \neq 0 \; at \; x = \frac{\pi}{2}, \frac{3\pi}{2} \in (0, 2\pi)$$

Hence the curve has point of inflexion at the points $\left(\frac{\pi}{2},\ 0\right)$ and $\left(\frac{3\pi}{2},\ 0\right)$.

Example 8: Find the point of inflexion on the curve $y = x^4 - 2x^3 + 2$.

Solutions: We have

$$y = x^4 - 2x^3 + 2$$

Differentiating the equation w.r.t. x we have

$$\frac{dy}{dx} = 4x^3 - 6x^2$$

Again differentiating the equation again w.r.t. x we have

$$\frac{d^2y}{dx^2} = 12x(x-1) = 0, when \; x = 0 \; or \; x = 1.$$

Again differentiating the equation again w.r.t. x we have

$$\frac{d^3y}{dx^3} = 24x - 12 \neq 0, when \; x = 0 \; or \; x = 1.$$

Hence the curve has point of inflexion at the points (0, 2) and (1, 1).

Example 9: Show that the curve $y = e^{-x^2}$ has only one point of inflexion.

Solutions: We have

$$y = e^{-x^2}$$

Differentiating the equation w.r.t. x we have

$$\frac{dy}{dx} = -2xe^{-x^2}$$

Again differentiating the equation again w.r.t. x we have

$$\frac{d^2y}{dx^2} = 4x^2 e^{-x^2} - 2xe^{-x^2} = 0 \, if \; x = 0 \; or \; \frac{1}{2}$$

Again differentiating the equation again w.r.t. x we have

$$\frac{d^3y}{dx^3} = 12xe^{-x^2} - 8x^3 e^{-x^2} \neq 0 \; at \; x = \frac{1}{2} \; only.$$

Hence the curve has only one point of inflexion.

I.Q. 6

I.Q. 7

I.Q. 8

I.Q. 9

I.Q. 10

5. Curve Sketching:

The object of curve sketching is to find the general appearance of a curve without plotting lots of points on the graph and avoiding laborious numerical calculations. For tracing a curve whose equation is given in the rectangular Cartesian coordinate system, that is, in terms of x and y, you must remember the following properties of curves.

1. If the equation f(x, y) = 0, remains unchanged when y is replaced by – y, i.e., f(x, y) = f(x, -y) = 0, then the curve is symmetrical with respect to the x-axis. e.g. $xy^2 = 4a^2(2a - x), y^2 = -4ax$.
2. If the equation f(x, y) = 0, remains unchanged when x is replaced by – x, i.e., f(x, y) = f(-x, y) = 0, then the curve is symmetrical with respect to the y-axis. e.g. $x^2 y = 4a^2(2a - y), x^2 = -4ay$.
3. If the equation f(x, y) = 0, remains unaltered when the signs of both x and y are replaced by their opposites, i.e., f(x, y) = f(-x, -y) = 0, the curve is symmetric about the origin that is, there is a symmetry in the opposite quadrants. e.g. $x^2 + y^2 = a^2, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
4. If the equation f(x, y) = 0, remains unaltered when x and y are interchanged, i.e., f(x, y) = f(y, x) = 0, then the curve is symmetric about the line y = x. e.g. $xy = 1, x^3 + y^3 = 3axy$.

The above four points will help you to find the symmetries of the curve.

After the given function has been tested for symmetries the following steps are to be performed.

1) Sometimes it is possible to find the values of x for which y are not defined (imaginary) i.e., no part of the graph for these values of x.

e.g. $y^2(2a-x)=x^3, or, y=\pm x\left[\frac{x}{(2a-x)}\right]^{1/2}$. We cannot have x > 2a or x < 0. Therefore, the graph must lie in the region bounded by the lines x = 2 a and x = 0.

2) If the equation does not contain a constant term, the curve passes through the origin. If the curve passes through the origin find the equation of the tangent at the origin by equating to zero the term of the lowest degree in the equation of the curve.
3) Find the points where the curve cuts the axes. Put x = 0 in the equation and solve the resulting equation to get the points where the curve cuts the y - axis. Similarly, put y = 0, solve resulting equation for x to get the points where the curve crosses x - axis.
4) Find those values of x for which y = 0 or y tends to infinity and the values of y for which values of x = 0 or x tends to infinity. This will give us the asymptotes parallel to the axes.
5) Also, find the oblique asymptotes to the given curve, if any.
6) Find the points of maxima and minima, range where the curve concave up and concave down, singular points to get an idea about the shape of the curve.

We will now take up a few simple standard cases of curve-tracing.

Example 10: Trace the curve $y=x^3$.

Solution: To trace the graph, we will use the following steps:

1. Since the equation of the curve remains unchanged when x and y are replaced by – x and – y, respectively, therefore the curve is symmetric about the origin. Also the curve is symmetrical in opposite quadrants.
2. Since, y is positive when x is positive and y is negative when x is negative, therefore, the curve lies in the 1st and 3rd quadrants only.

3. The curve passes through the origin and the equation of tangent at origin is y = 0.
4. The curve meets the coordinate axes only at the origin.
5. There are no asymptotes.
6. $y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2 = 0 \; at \; x = 0$, but does not change sign as x passes through 0, therefore, y does not possess any extreme value. But, $\frac{d^2y}{dx^2} = 6x = 0 \; and \; \frac{d^3y}{dx^3} = 6 \neq 0 \; at \; x = 0$ therefore, there is a point of inflexion at x = 0. Thus (0, 0) is a point of inflexion.
7. When x = 0 1 2 3 ∞

   y = 0 1 8 27 ∞

   The approximate shape of the curve is shown in the Fig 4.

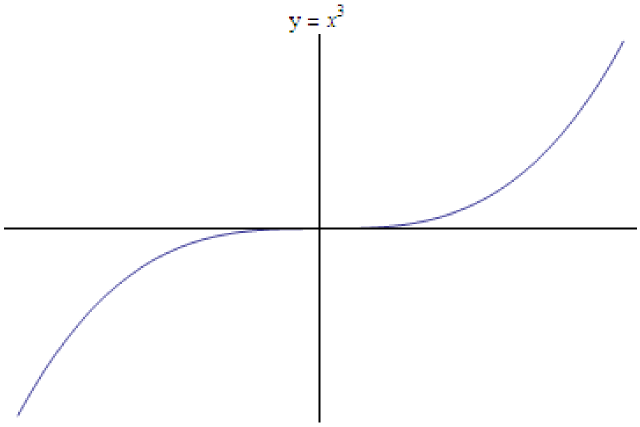


Fig 4: The Curve $y = x^3$

Value Addition:
The graph of any polynomial of degree n has at most n x–intercepts, at most n–1 relative extrema, and at most n–2 points of inflexion.

Example 11: Sketch the curve $y = \frac{x^2+1}{x^2-1}$.

Solution: To sketch the graph, we use the following steps:

1. Since the equation of the curve remains unchanged when x is replaced by – x, therefore the curve is symmetrical with respect to the y-axis.

2. When x lies between 0 and 1, y is negative therefore the curve lies in 4^{th} quadrant for the interval (0, 1). When x is greater than +1, y is positive therefore the curve lies in 1^{st} quadrant for the interval $(1, \infty)$. When x lies between -1 and 0, y is negative therefore the curve lies in 3^{rd} quadrant for the interval (-1, 0). When x is less than -1, y is positive therefore the curve lies in 2^{nd} quadrant for the interval $(-\infty, -1)$.
3. The curve does not pass through origin.
4. When x = 0, y = -1 and when y = 0, x is imaginary. The curve meets the coordinate axes only at (0, -1).
5. Asymptotes parallel to the x-axis is y = 1; and parallel to the y-axis are x =± 1.
6. $y=\frac{x^2+1}{x^2-1} \Rightarrow \frac{dy}{dx}=\frac{2x}{x^2+1}-\frac{2x(x^2+1)}{(x^2+1)^2}=0, \frac{d^2y}{dx^2}=-4\neq 0\, at\, x=0$, therefore, the curve is concave at x = 0.
7. Special points

x =	0	1/2	1	3	∞
y =	-1	-5/3	∞	5/4	1

The approximate shape of the curve is shown in Fig 5.

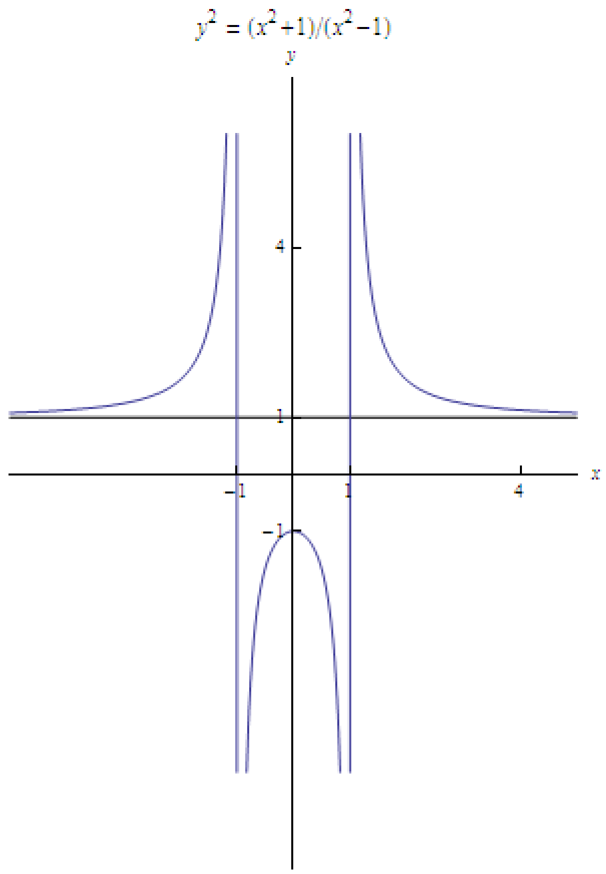


Fig 5: The Curve $y = \dfrac{x^2+1}{x^2-1}$

Example 12: Sketch the graph $y^2 = (x-1)(x-2)(x-4)$.

Solutions: To sketch the graph, we use the following steps:

1. Since the equation of the curve remains unchanged when y is replaced by − y, therefore the curve is symmetric about the x - axis.
2. When x < 1, y^2 is negative, so, to the left of the line x = 1, no part of the curve lies. i.e., no part of the curve lies in the 2nd and 3rd quadrants. Also, since y^2 is negative if 2 < x < 4, no part of the curve lies in the region bounded by the lines x = 2 and x = 4.
3. The curve meets the x − axis at the points (1, 0), (2, 0), (4, 0) and it does not meet the y − axis.
4. There are no asymptotes.

5. $y^2 = (x-1)(x-2)(x-4) \Rightarrow \frac{dy}{dx} = 0 \; when \; x = \frac{1}{3}\left(7 \pm \sqrt{7}\right)$, y^2 is maximum at $x = \frac{1}{3}\left(7 - \sqrt{7}\right)$. Hence, the portion of the curve is an oval between x = 1 and x = 2.
6. When x > 4, y^2 keeps on increasing and tends to ∞ as x tends to ∞.

   The approximate shape of the curve is shown in Fig 6.

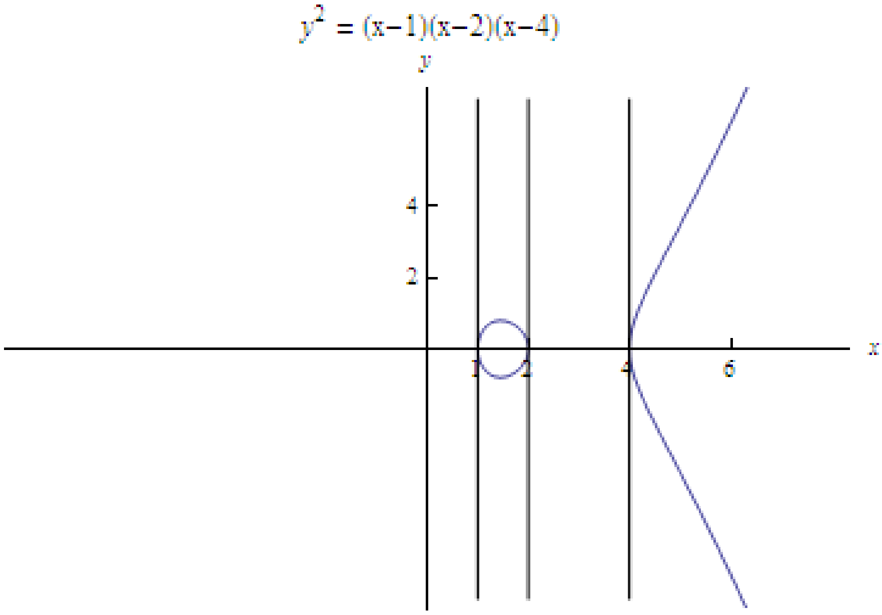


Fig 6: The Curve $y^2 = (x-1)(x-2)(x-4)$

Example 13: Generate or sketch the graph of $y^2 = (x-2)^3$

Solution: To sketch the graph, we use the following steps:

1. Since the equation of the curve remains unchanged when y is replaced by – y, therefore the curve is symmetric about the x - axis.
2. When x < 2, y^2 is negative, so, to the left of the ordinate x = 2, no part of the curve lies. i.e., no part of the curve lies in the 2nd and 3rd quadrants.
3. The curve meets the x – axis at the point (2, 0) and it does not meets the y – axis. The tangent at (2, 0) are given by $y^2 = 0$, hence (2, 0) is a cusp.
4. There are no asymptotes.

5. $y^2=(x-2)^3 \Rightarrow 2y\frac{dy}{dx}=3(x-2)^2$, so if $y>0, then \frac{dy}{dx}>0$ and if $y<0, then \frac{dy}{dx}<0$, i.e., in the 1st quadrant y increases as x increases, and in the 4th quadrant y decreases as x increases

6. For the branch $y^2=(x-2)^3, y\to+\infty \, as \, x\to+\infty$ and for the branch $y^2=-(x-2)^3, y\to-\infty \, as \, x\to+\infty$

The approximate shape of the curve is shown in the figure below

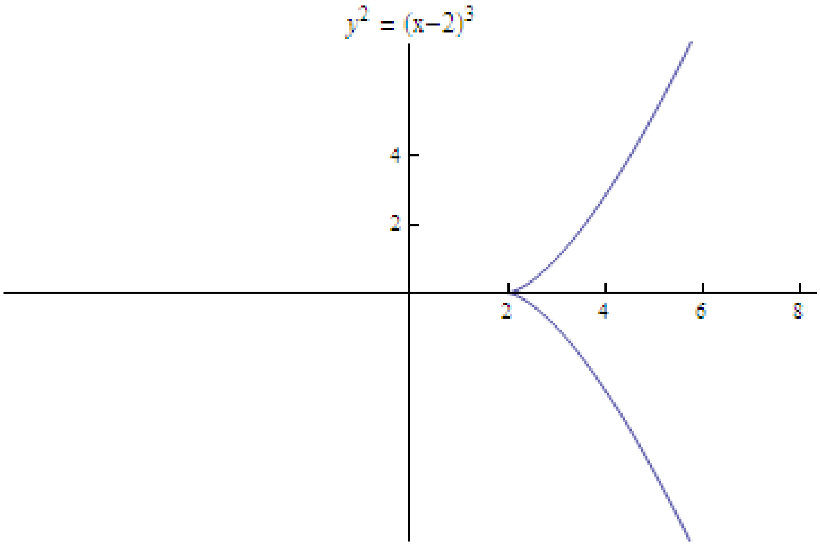


Fig 7: The Curve $y^2=(x-2)^3$

Example 14: Generate the graph of $y^2(a+x)=x^2(a-x)$.

Solutions: To sketch the graph, we use the following steps:

1. Since the equation of the curve remains unchanged when y is replaced by – y, therefore the curve is symmetric about the x - axis.

2. Since $y=\pm x\sqrt{\left[\left\{(a-x)/(a+x)\right\}\right]}$, y is real when – a < x < a, and for other values of x, y is imaginary.

3. The curve passes through the origin and the equation of tangents at origin are y = ± x.

4. The curve meets the coordinate axes only at the points (0, 0) and (a, 0).

5. Asymptotes parallel to y – axis is x = - a.

6. Special points are

When x =	-a	-a/2	0	a
y =	∞	$-a\sqrt{3/2}$	0	0

The approximate shape of the curve is shown in the figure below

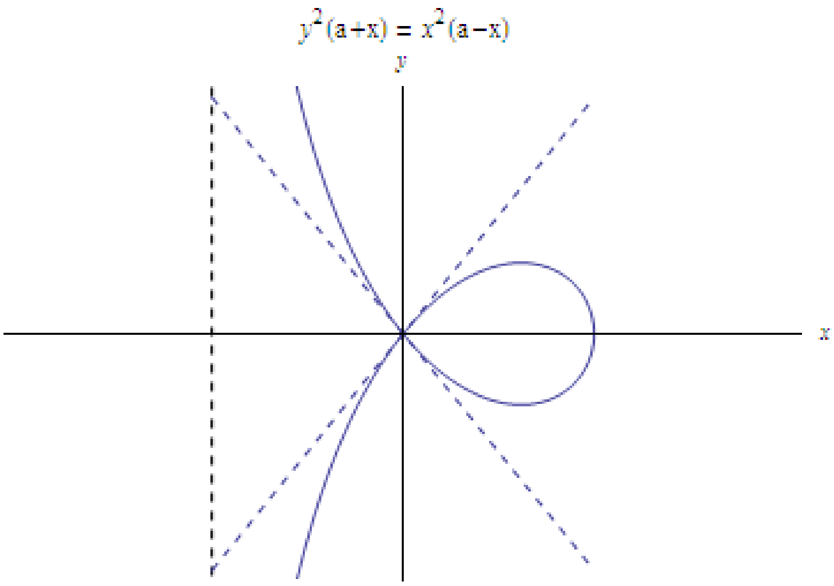


Fig 8: The Curve $y^2(a+x)=x^2(a-x)$

Example 15: Trace the curve $y^2(x-a)=x^2(x+a), a>0$.

Solutions: To sketch the graph, we use the following steps:

1. Since the equation of the curve remains unchanged when y is replaced by – y, therefore the curve is symmetric about the x - axis.

2. Since $y=\pm x\sqrt{\left[\{(x+a)/(x-a)\}\right]}$, $y^2 < 0$ when – a < x < a, and x ≠ 0, so no part of the graph lies in the region bounded by the ordinates x = ± a, except the origin.

3. The curve passes through the origin and the equation of tangents at origin are given by $x^2 + y^2 = 0$. Since the tangent at origin is imaginary, hence, origin is a conjugate point. The tangent at (-a, 0) is x + a = 0.
4. The curve meets the coordinate axes only at the points (0, 0) and (-a, 0).
5. Asymptotes parallel to y – axis is x = a and y = ± (x + a) are the oblique asymptotes.

6. Special points are

When	x = -a	0	a
	y = 0	0	∞

The approximate shape of the curve is shown in the figure below

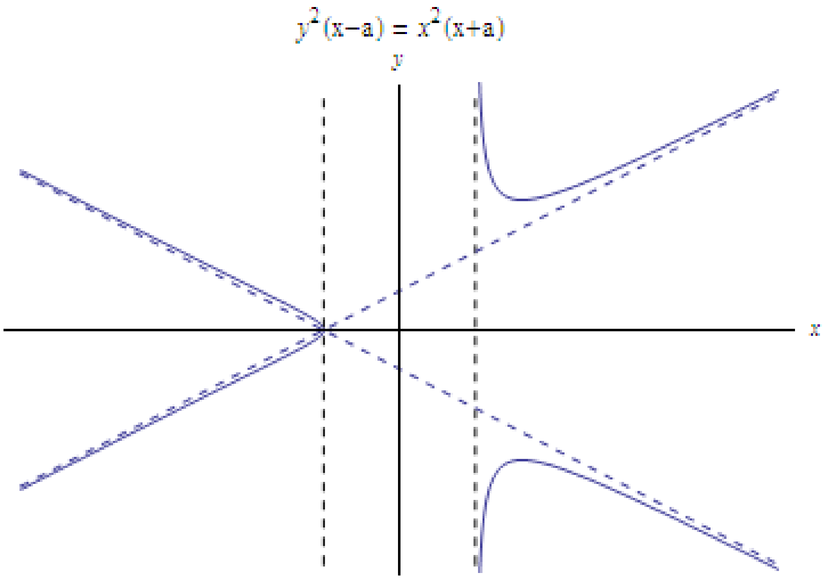


Fig 9: The Curve $y^2(x-a)=x^2(x+a)$

Example 16: Sketch the graph of $y^2(2a-x)=x^3$.

Solutions: To sketch the graph, we use the following steps:

1. Since the equation of the curve remains unchanged when y is replaced by – y, therefore the curve is symmetric about the x - axis.
2. When x > 2a, y is imaginary, that is, the curve does not exist for values of x > 2a. Similarly, the curve does not exist for negative values of x.
3. The curve passes through the origin.
4. The tangent at the origin is given by y = 0. Thus, the x-axis is the tangent at the origin.
5. The asymptote is given by 2a – x = 0.
6. $\frac{dy}{dx} > 0 \ for \ x > a$. Therefore, the function is increasing in the interval $]0, a[$.
7. When x tends to 2a, y tends to ∞.

   The approximate shape of the curve is shown in Fig 10.

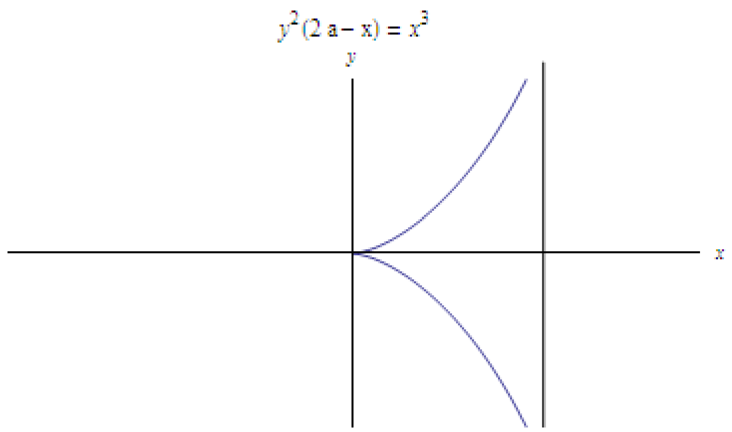


Fig 10: The Curve $y^2(2a - x) = x^3$

Example 17: Sketch the graph of $y = e^x$.

Solutions: To sketch the graph, we use the following steps:

1. The curve is not symmetric.
2. The curve meets the y – axis at the point (0, 1).

3. There are no asymptotes.

4. Since

$$y\frac{d^2y}{dx^2}=e^x.e^x=\left(e^x\right)^2>0$$

$y\dfrac{d^2y}{dx^2}$ is positive for all values of x, the curve is at every point convex to the foot of the corresponding ordinate.

5. As x increases from - ∞ to ∞, y increases from 0 to ∞.

The approximate shape of the curve is shown in Fig 11.

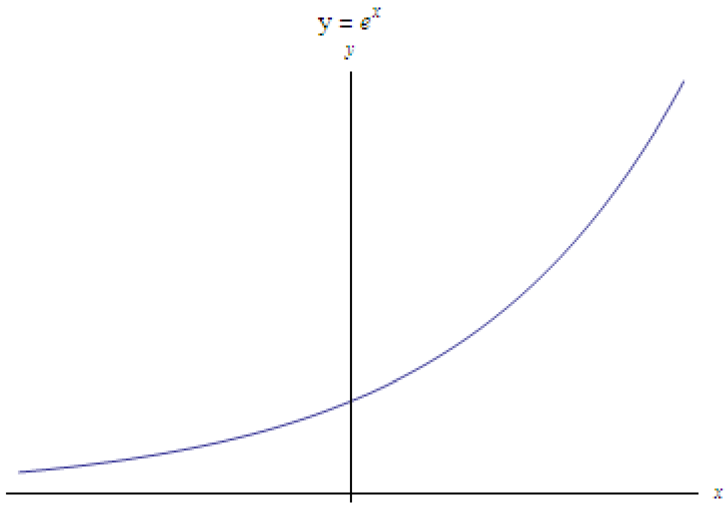


Fig 11: The Curve $y=e^x$

Example 18: Sketch the graph of the curve $y=x\left(x^2-3\right)$.

Solutions: To sketch the graph, we use the following steps:

1. The curve is not symmetric.

2. The origin lies in the curve and the tangent at the origin is $y-3x=0$

3. The curve meets the x – axis at the points $(0,\ 0)$, $\left(-\sqrt{3},\ 0\right)$, $\left(\sqrt{3},\ 0\right)$ and meets the y – axis only at the origin.

4. There are no asymptotes.

5. $y = x\left(x^2 - 3\right) \Rightarrow \dfrac{dy}{dx} = 3x^2 - 3\, when\, x = \pm 1$, y is maximum at $x = -1$ and y is minimum at $x = 1$.

6. The curve is increasing in the interval $(-\infty, -1) \cup (1, \infty)$ and decreasing in the interval $(-1, 1)$.

The approximate shape of the curve is shown in Fig 12.

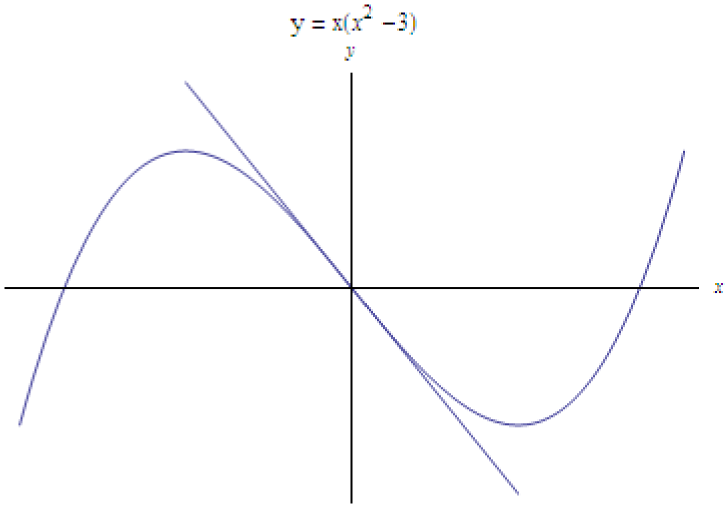


Fig 12: The Curve $y = x\left(x^2 - 3\right)$

Example 19: Sketch the graph of the curve $y = (x-2)(x+1)^2$.

Solutions: To sketch the graph, we use the following steps:

1. The curve is not symmetric.
2. The origin does not lie on the curve.
3. The curve meets the x – axis at the points (-1, 0), (2, 0) and it meets the y – axis at the point (0, -2).
4. There are no asymptotes.
5. $y = (x-2)(x+1)^2 \Rightarrow \dfrac{dy}{dx} = 3x^2 - 3 = 0\, when\, x = \pm 1$. y is maximum at $x = -1$ and y is minimum at $x = 1$.
6. The curve is increasing in the interval $(-\infty, -1) \cup (1, \infty)$ and decreasing in the interval $(-1, 1)$.

The approximate shape of the curve is shown in Fig 13.

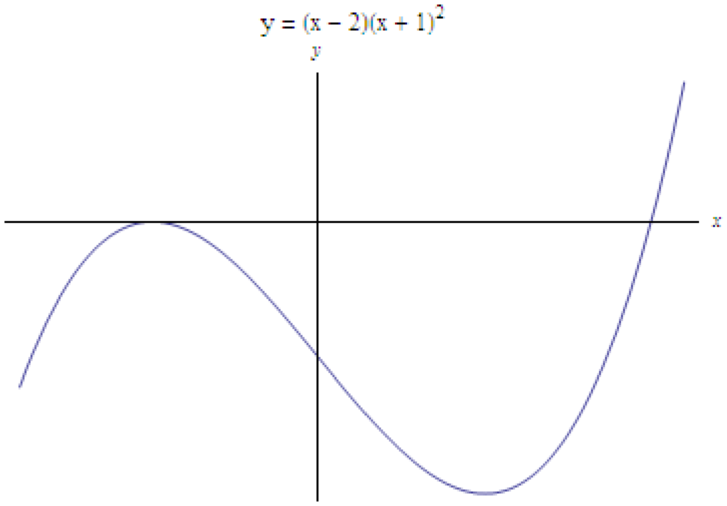


Fig 13: The Curve $y=(x-2)(x+1)^2$

Example 20: Sketch the graph of curve $y=x^2(x-3a), a>0.$.

Solutions: To sketch the graph, we use the following steps:

1. The curve is not symmetric.
2. When x < 3a, y is negative, so, no part of the curve lies in the 2nd quadrant.
3. The curve meets the x – axis at the points (0, 0), (3 a, 0) and it meets the y – axis at the origin.
4. The tangent at the origin is y = 0. Since y < 0 for small values of x, hence the shape of the graph in the neighbourhood of the origin is as shown in the Fig. 14.
5. There are no asymptotes.
6. $y=x^2(x-3a)\Rightarrow\dfrac{dy}{dx}=3x(x-2a)>0\, when\, x<0\, and\, in\,]2a,\infty[$

   $$<0\, in\,]0,2a[$$

   So y is increasing in $]-\infty,0[\cup]2a,\infty[$ and decreasing in $]0,2a[$. Also y has a maximum at $(0,0)$ and minimum at $(2a,-4a^3)$.
7. y tends to -∞ as x tends to -∞ and y tends to +∞ as x tends to +∞.

The approximate shape of the curve is shown in Fig 14.

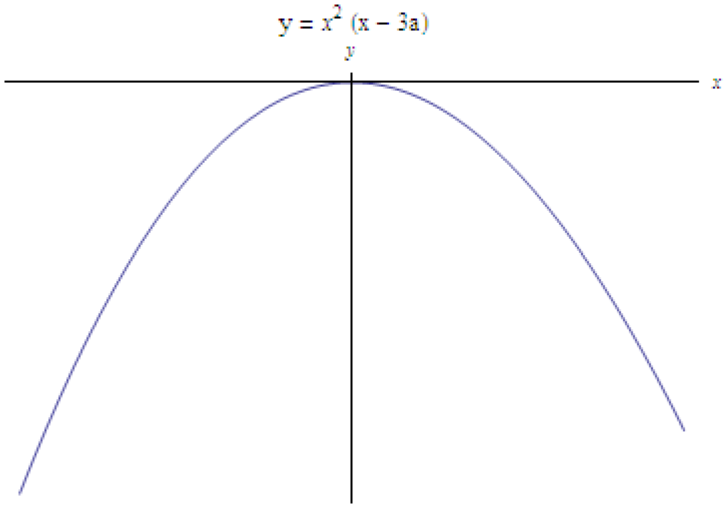


Fig 14: The Curve $y = x^2(x-3a)$

I.Q. 11

I.Q. 12

I.Q. 13

I.Q. 14

I.Q. 15

I.Q. 16

I.Q. 17

I.Q. 18

I.Q. 19

I.Q. 20

Exercise:

1. Find the range of values of x for which the curve $y = x^3 - 6x^2 + 11x - 6$ is concave upwards or downwards.
2. Examine the curve $y = \sin x$ for convexity and concavity in the interval $(0, 2\pi)$.
3. Examine the curve $y = x^4 - 6x^3 + 12x^2 + 5x + 7$ for convexity and concavity. Also determine its point of inflexion.

4. Show that the curve $y=x^3$, traced in example 5 has a point of inflexion at the origin.
5. Determine the points of inflexion of the curve $y^2=(x-a)^2(x-b)$
6. Sketch the following Cartesian curves:

   a) $y\left(1+x^2\right)=x^2$

   b) $y\left(x^2-1\right)=x$

   c) $x^2y^2=x^2-a^2$

   d) $x^2\left(x^2+y^2\right)=a^2\left(x^2-y^2\right)$

   e) $\left(x^2-a^2\right)\left(y^2-b^2\right)=a^2b^2$

   f) $y^3=a^2x-x^3$

   g) $a\,y^2=x^2(x-a)$

   h) $x(x-2a)\,y^2=a^2(x-a)(x-3a)$

   i) $y^2\left(x^2+y^2\right)+a^2\left(x^2-y^2\right)=0$

   j) $a^{3/2}y=(x-a)^2\sqrt{(x-b)}, a>b.$

Summary:

We now end this chapter by giving a summary of it. In this chapter we have covered the following

(1) Role of second derivative $\frac{d^2y}{dx^2}$ in determining the behaviour of the curve y = f(x) in a given domain.

(2) Obtain range for which the curve is concave up or concave down.

(3) Method to determine concavity and convexity of a given function.

(4) Method to determine point of inflexion of a given function.

(5) Method of tracing some Cartesian curves.

Glossary:

A Asymptotes: A straight line is said to be an asymptote of the curve if as the point P on the curve tends to infinity along the curve, the perpendicular distance of P from the straight line tends to zero.

C Concave up: On interval I where the graph of the function has upward curvature.

Concave down: On interval I where the graph of the function has downward curvature.

P Points of inflexion: Those points where $f''(x)=0,$ and changes sign, the curve changes the direction of its concavity and crosses its tangent.

S Symmetry:

- If f(x, y) = f(x, -y) , then the curve is symmetrical with respect to the x-axis.
- If f(x, y) = f(-x, y) , then the curve is symmetrical with respect to the y-axis.
- If f(x, y) = f(-x, -y), the curve is symmetric about the origin that is, there is a symmetry in the opposite quadrants.
- If f(x, y) = f(y, x), then the curve is symmetric about the line y = x.

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